Solution of the WoLLIC 2005 Problem

There are 400 Theories with 4 Propositional Symbols

I would like to thank very much all the participants. We had two correct solutions to the WoLLIC 2005 Problem.

First Winner

Prof. Dr. Alasdair Urquhart
Contact: http://www.cs.toronto.edu/DCS/People/Faculty/urquhart.html
Photo: http://andrej.com/mathematicians/U/Urquhart_Alasdair.html
Bibliography: http://sun3.lib.uci.edu/eyeghiay/Philosophy/Colloquia/urquhart.html pointed to the solution obtained by George Polya http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Polya.html http://www.educ.fc.ul.pt/docentes/opombo/seminario/polya/biografia.htm author of the famous book http://www.cis.usouthal.edu/misc/polya.html*How to solve it*

A copy of the paper can be obtained here.

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Date: Sat, 2 Jul 2005 10:57:58 -0400
From: Alasdair Urquhart <urquhart@cs.toronto.edu>
Subject: WoLLIC'05 problem

A theory in a finite number of propositional symbols
can be axiomatized by a single axiom, and any two
such theories are logically equivalent if and only if
their single axioms are logically equivalent.
The problem is therefore equivalent to asking how
many non-equivalent formulas there are in 4 variables,
excluding tautologies and contradictions.

This problem was solved by George Polya (Journal of
Symbolic Logic Volume 5 (1940), pp. 98-103).  Polya showed
that there are 402 distinct Boolean formulas with respect to
the group of permutations and complementations.
The answer to the problem is therefore 400.

More extensive computations are in Michael Harrison's
book "Introduction to Switching and Automata Theory."

Alasdair Urquhart
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Second Winner

Prof. Dr. Vaughan Pratt
http://boole.stanford.edu/pratt.html sent us a very interesting solution in the form of a small C program the calculates the correct answers. He also kindly sent an explanation of how the program works.

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Date: Tue, 12 Jul 2005 23:00:52 -0700
From: Vaughan Pratt <pratt@cs.stanford.edu>
Subject: WoLLIC'05 problem

Dear Guilherme,

Thanks for your clarification of the problem.  The attached program
reports 400 theories.

Best regards,

Vaughan Pratt
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Motivation

I found the problem when I was looking for a unique syntactical representation for propositional theories to be used in Belief Change operations. This led me the Prime Implicants/Implicates representation and from there to the observation that they can be classified in equivalence classes if we ignore the specific identities and signs of the propositional symbols.

More details of how these classes were calculated can be found here.

A complete list of the 221 prime implicants/implicates pairs that represent the 400 possible theories with 4 propositional symbols, together with some interesting properties, can be found here.

Conclusion

To my delight and surprise the problem proved to be relevant to pure *Logic*, as George Polya's pre-computational era solution shows, and to *Information and Computation*, as the applications of Prof. Pratt solution in the characterization of the power of Vector Machines and in the implementation of Sun's Pixrect graphics interface shows.

To complete the WoLLIC acronym we need *Language*. I'll try to fill the gap. In his 1922 *Tractatus Logico-Philosophicus*, the "young" Wittgenstein states that philosophical problems arise because of misuse of language, and it is only when we understand everyday language that these questions may be dealt with:

> (...) The whole sense of the book might be summed up the following words: what can be said at all can be said clearly, and what we cannot talk about we must pass over in silence. Thus the aim of the book is to draw a limit to thought, or rather – not to thought, but to the expression of thoughts: for in order to be able to draw a limit to thought, we should have to find both sides of the limit thinkable (i.e. we should have to be able to think what cannot be thought). It will therefore only be in language that the limit can be drawn, and what lies on the other side of the limit will simply be nonsense. (...)

Im this context, if we lived in a 4 bits world and if we only had propositional logic as a language, all that could be "said clearly" would be those 400 theories and "Wovon man nicht sprechen kann, darüber muSS man schweigen." (Whereof one cannot speak, thereof one must be silent.)